Electric Potential of a Torus Knot Along the Axis

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Particularly, this means that we need a parametrization of a knot to do calculations, such as through numerical methods or **complex analysis**.

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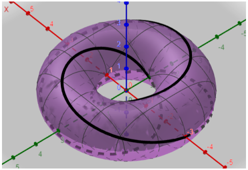

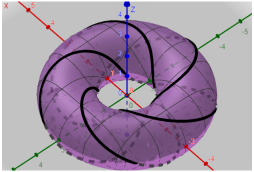

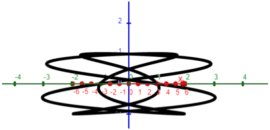

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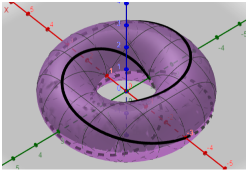

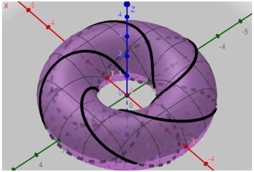

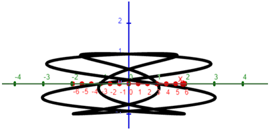

Exercise to the viewer: rotating the last one looks really cool.

Parametrizations of Torus Knots

A (p, q)-torus knot can be parametrized by $r(t) = (x(t), y(t), z(t))$ with

$$x = (\cos(qt) + 2)\cos(pt)$$
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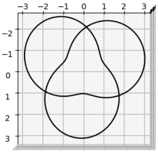
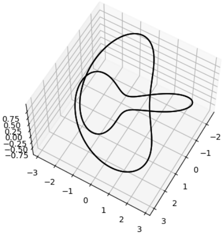
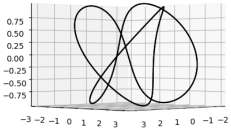
Here's the simplest non-trivial knot, the $(2, 3)$-torus knot:







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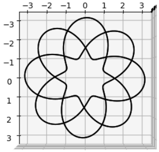
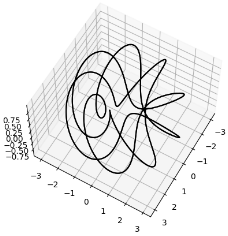
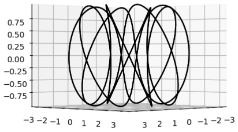
Here's a more complicated knot, the $(3, 8)$-torus knot:







Electric Potential and Electric Field

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- Thus, we may define the electric potential of a curve $r(t)$ to be

$$\Phi(x) = \int_{t=0}^{2\pi} \frac{1}{|x - r(t)|} dr = \int_0^{2\pi} \frac{|r'(t)|}{|x - r(t)|} dt.$$

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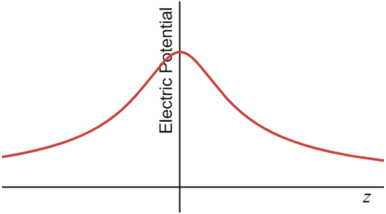
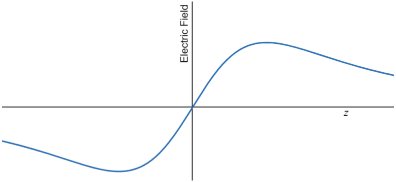
- The electric field is the gradient of the electric potential and is a vector. It is

$$\nabla\Phi(x) = \int_0^{2\pi} \frac{(x - r(t))}{|x - r(t)|^3} |r'(t)| dt.$$

Graphs of the Electric Potential and Electric Field

To get an idea of what these functions look like, here are some graphs. Our goal is to use actual calculations to show these graphs are right.





Observations on the Electric Potential and Field

Now, we go onto the z-axis. We can now let $x = (0, 0, \alpha)$. Plugging in our parametrization for $r(t)$ and doing some substitutions gives us:

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$$|r'(t)| = \sqrt{q^2 + p^2(2+\cos(u))^2}$$

$$\Phi((0,0,\alpha)) = \int_0^{2\pi} \frac{\sqrt{q^2 + p^2(2+\cos(u))^2}}{\sqrt{\alpha^2 + 2\alpha\sin(u) + 5 + 4\cos(u)}} du$$

$$\nabla\Phi((0,0,\alpha))_z = \int_0^{2\pi} \frac{(\alpha + \sin(u))\sqrt{q^2 + p^2(2+\cos(u))^2}}{(\alpha^2 + 2\alpha\sin(u) + 5 + 4\cos(u))^{3/2}} du$$

Observe that the numerator depends on p and q, so we may scale p and q while scaling the integral by the same factor.

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We can get some results:

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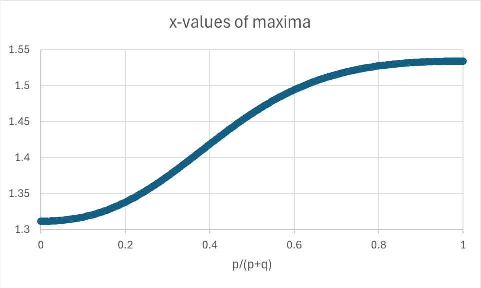
We can also look at the electric field's extreme points:



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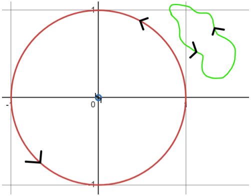
For example, if we integrate the function $\frac{1}{x}$ along these three contours, the red and blue are both $2\pi i$ and the green is 0.



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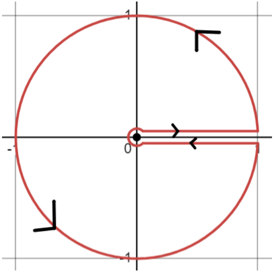

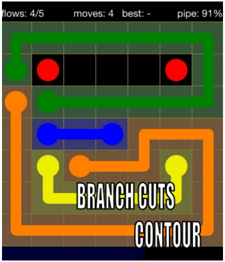

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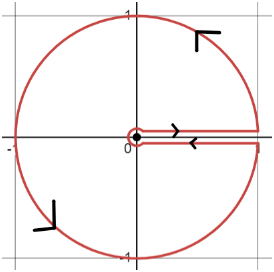

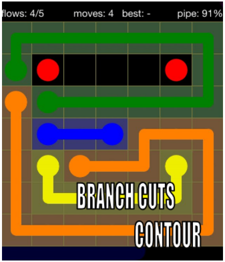

Note that, the two segments don't cancel!

Approximating the Length with Complex Analysis

Now, let's take something like

$$\int_0^{2\pi} |r'(t)| dt = \int_0^{2\pi} \sqrt{q^2 + p^2(2 + \cos(u))^2} du$$

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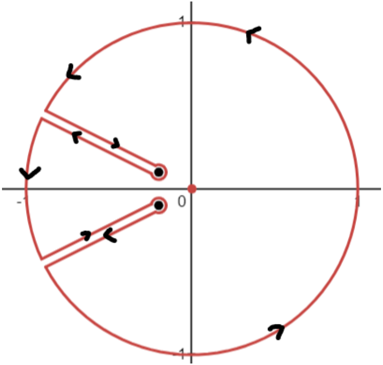
- The ω^2 means 0 and ∞ are not branch points.

Approximating the Length with Complex Analysis

Does this contour work?



Approximating the Length with Complex Analysis

We now have

$$\int_0^{2\pi} |r'(t)| dt = 8\pi + 2\int_{-n}^{n} \frac{\sqrt{q^2 + p^2 \left(2 + \frac{m+xi}{2} + \frac{1}{2(m+xi)}\right)^2}}{m+xi} dx$$

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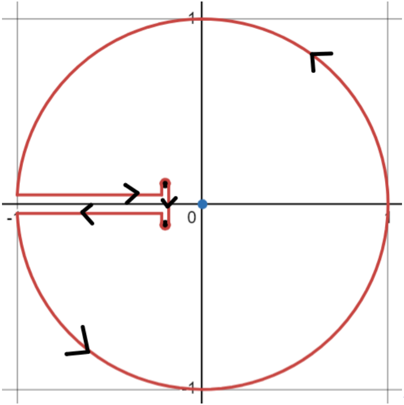
We turned it into the integral along a segment!



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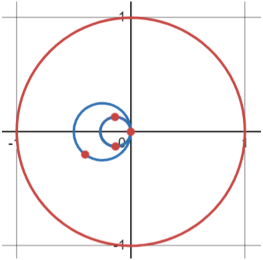
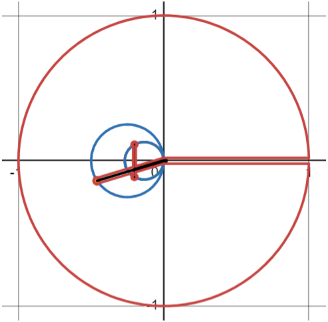
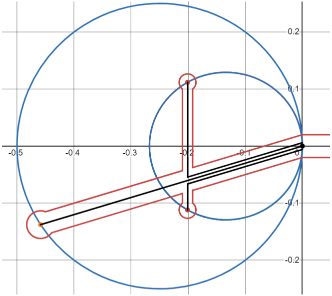
- The electric potential has 8 branch points. Two are $0, \infty$, four are the same as before, and two new ones come from the denominator. They are at $-2 - \alpha i$ and $\frac{-2-\alpha i}{4+\alpha^2}$. Conveniently, the branch points lie on ellipses independent of p, q, α.



The Electric Potential Contour Integral

We can thus approximate the integral using integrals on segments between the points!





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